A Novel Error-Reducing Methodology on the Fast Fourier Transform Option Valuation

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Agenda

Introduction

- The BS formula is inaccurate $\Rightarrow$ We need other underlying assumptions.
- Jump-diffusion models, VG, CGMY, Heston, etc.
- Complicated densities but simple characteristic functions.
- Carr and Madan (1999)
  - Characteristic functions ~~$\Rightarrow$ Density~~ $\Rightarrow$ Option Prices.
  - Computational power of the FFT.
  - Simultaneously price N otherwise identical vanilla options with different strike prices.
  - 1 FFT v.s. 1 FFT plus 1 convolution.
- Highly oscillatory integrand $\Rightarrow$ Considerable grid points and computational time are needed.
- Negative prices. (see e.g. Carr and Madan (2009))
- Improve pricing efficiency and alleviate the negative prices problem.

Carr and Madan's Option Pricing Method

- Carr and Madan (1999) suggest that

$$C_T(k) = \frac{e^{-dk}}{\pi} \int_0^{\infty} e^{-ivk} \psi_T(v) \, dv,$$

where $d > 0$;

$$\psi_T(v) = \frac{e^{-rT} \, \phi_T(v - (d+1)i)}{d^2 + d - v^2 + i(2d+1)v};$$

$\phi_T(v)$ is the risk-neutralized characteristic function of $\ln S_T$.

- $$C_T(k) \approx \frac{e^{-dk}}{\pi} \sum_{j=1}^{N} e^{-iv_j k} \psi_T(v_j) \eta.$$

Carr and Madan's Option Pricing Method

- FFT:

$$w(k) = \sum_{j=1}^{N} e^{-i\frac{2\pi}{N}(j-1)(k-1)}\chi(j) \quad \text{for} \quad k = 1, 2, \ldots, N.$$

- $O(N^2) \Rightarrow O(N \log_2 N)$.
- Let $k_u = k_1 + \Lambda(u-1) \quad \text{for} \quad u = 1, 2, 3, \ldots, N$, then

$$\begin{aligned}
C_T(k_u) &\approx \frac{e^{-dk_u}}{\pi} \sum_{j=1}^{N} e^{-iv_j(k_1+\Lambda(u-1))}\psi_T(v_j)\eta \\
&= \frac{e^{-dk_u}}{\pi} \sum_{j=1}^{N} e^{-i\Lambda\eta(j-1)(u-1)} e^{-ik_1 v_j}\psi_T(v_j)\eta. \\
(\text{Let } \Lambda\eta = \frac{2\pi}{N}) \quad &= \frac{e^{-dk_u}}{\pi} \sum_{j=1}^{N} e^{-i\frac{2\pi}{N}(j-1)(u-1)} e^{-ik_1 v_j}\psi_T(v_j)\eta.
\end{aligned}$$

Our Novel Error-Reducing Method

- $\psi_T(v)$: Fourier transform of call price under a risk-neutralized target density.
- $$\begin{aligned} C_T(k) &= \frac{e^{-dk}}{\pi}\int_0^\infty e^{-ivk}\psi_T(v)\,dv \\ &= \frac{e^{-dk}}{\pi}\int_0^\infty e^{-ivk}[\psi_T^{\text{proxy}}(v)+\psi_T^{\text{residual}}(v)]\,dv \\ &= \underbrace{C_T^{\text{proxy}}(k)}_{\text{Analytic}} + \underbrace{C_T^{\text{residual}}(k)}_{\text{Numerical Integration}} . \end{aligned}$$
- Which "proxy" should we choose?

Merton's Jump-Diffusion Model

- $$dS_t = \left(\mu + \frac{\sigma^2}{2}\right) S_t\, dt + \sigma S_t\, dW_t + (e^{J_t} - 1) S_t\, dM_t,$$

  where W_t is a standard Brownian motion; M_t is a Poisson process with jump intensity λ; $J_t \overset{iid}{\sim} N(\alpha, \beta^2)$.
- The discount rate implied by the SDE above is equal to

  $$\mu + \frac{\sigma^2}{2} + \lambda(e^{\alpha + \frac{\beta^2}{2}} - 1)$$

  since

  $$\begin{aligned} \mathbb{E}_{\mathbb{P}}[S_T] &= \mathbb{E}_{\mathbb{P}}\left[S_0\ e^{\mu T + \sigma W_T + \sum_{k=0}^{M_T} J_k}\right] \\ &= S_0\ e^{\mu T}\ e^{\frac{\sigma^2}{2} T}\ e^{\lambda T (e^{\alpha + \frac{\beta^2}{2}} - 1)}. \end{aligned}$$

Merton's Jump-Diffusion Model

- $$\phi_T^{\mathrm{MJD}}(u) = e^{-\lambda T + iu(\mu T + \ln S_0) - \frac{1}{2}\sigma^2 T u^2} \sum_{j=0}^{\infty} \frac{(\lambda T e^{iu\alpha - \frac{\beta^2 u^2}{2}})^j}{j!}$$
$$= \exp\left(iu \ln S_0 + \left(iu\mu - \frac{\sigma^2 u^2}{2} + \lambda \left(e^{i\alpha u - \frac{\beta^2 u^2}{2}} - 1 \right) \right) T \right).$$

- $$\begin{aligned}
k^{\text{th}}\text{cumulant} &:= (-i)^k (\log \phi_T^{\mathrm{MJD}})^{(k)}(0). \\
1^{\text{st}}\text{cumulant} &= \ln S_0 + (\mu + \lambda\alpha)\, T, \\
2^{\text{nd}}\text{cumulant} &= \left(\sigma^2 + \lambda\left(\alpha^2 + \beta^2\right)\right) T, \\
3^{\text{rd}}\text{cumulant} &= \lambda(\alpha^3 + 3\alpha\beta^2) T, \\
4^{\text{th}}\text{cumulant} &= \lambda(\alpha^4 + 6\alpha^2\beta^2 + 3\beta^4) T, \\
5^{\text{th}}\text{cumulant} &= \lambda(\alpha^5 + 10\alpha^3\beta^2 + 15\alpha\beta^4) T.
\end{aligned}$$

Merton's Jump-Diffusion Model

- $$C_T^{\mathrm{MJD}}(k) = \sum_{j=0}^{\infty} \frac{e^{-\lambda' T}(\lambda' T)^j}{j!} C_T^{\mathrm{BS}}(S_0, k, \sigma_j, r_j),$$

  where C_T^{BS} denotes the Black-Scholes call option pricing formula with the initial asset price S_0, the strike price $K := e^k$, the risk-free rate r_j and the volatility σ_j.

- $$\begin{aligned} \lambda' &:= \lambda e^{\alpha + \frac{\beta^2}{2}}, \\ \sigma_j &:= \sqrt{\sigma^2 + \frac{j\beta^2}{T}}, \\ r_j &:= \mu + \frac{\sigma^2}{2} + \frac{j(\alpha + \frac{\beta^2}{2})}{T}. \end{aligned}$$

Our Novel Error-Reducing Method

- $$\phi_T^{\text{MJD}}(u) = e^{-\lambda T + iu(\mu T + \ln S_0) - \frac{1}{2}\sigma^2 T u^2} \sum_{j=0}^{\infty} \frac{(\lambda T e^{iu\alpha - \frac{\beta^2 u^2}{2}})^j}{j!}.$$
- Let H be a positive integer.

$$\phi_T^{\text{MJD, 1}}(u) := e^{-\lambda T + iu(\mu T + \ln S_0) - \frac{1}{2}\sigma^2 T u^2} \sum_{j=0}^{H-1} \frac{(\lambda T e^{iu\alpha - \frac{\beta^2 u^2}{2}})^j}{j!},$$

$$\phi_T^{\text{MJD, 2}}(u) := e^{-\lambda T + iu(\mu T + \ln S_0) - \frac{1}{2}\sigma^2 T u^2} \sum_{j=H}^{\infty} \frac{(\lambda T e^{iu\alpha - \frac{\beta^2 u^2}{2}})^j}{j!}.$$

Our Novel Error-Reducing Method

- $$\psi_T^{\text{MJD}}(v) = \frac{e^{-(\mu+\frac{\sigma^2}{2}+\lambda(e^{\alpha+\frac{\beta^2}{2}}-1))T}\,\phi_T^{MJD}(v-(d+1)i)}{d^2+d-v^2+i(2d+1)v}.$$

- $$\psi_T^{\text{MJD, 1}}(v) := \frac{e^{-(\mu+\frac{\sigma^2}{2}+\lambda(e^{\alpha+\frac{\beta^2}{2}}-1))T}\,\phi_T^{\text{MJD, 1}}(v-(d+1)i)}{d^2+d-v^2+i(2d+1)v},$$
$$\psi_T^{\text{MJD, 2}}(v) := \frac{e^{-(\mu+\frac{\sigma^2}{2}+\lambda(e^{\alpha+\frac{\beta^2}{2}}-1))T}\,\phi_T^{\text{MJD, 2}}(v-(d+1)i)}{d^2+d-v^2+i(2d+1)v}.$$

Our Novel Error-Reducing Method

$$
\begin{aligned}
C_T(k) &= \frac{e^{-dk}}{\pi}\int_0^\infty e^{-ivk}\psi_T(v)\,dv \\
&= \frac{e^{-dk}}{\pi}\int_0^\infty e^{-ivk}[\psi_T^{\text{MJD}}(v)+\psi_T^{\text{RESIDUAL}}(v)]\,dv \\
&= \frac{e^{-dk}}{\pi}\int_0^\infty e^{-ivk}\left[\psi_T^{\text{MJD,1}}(v)+\psi_T^{\text{MJD, 2}}(v)\right]\,dv \\
&\quad+\frac{e^{-dk}}{\pi}\int_0^\infty e^{-ivk}\psi_T^{\text{RESIDUAL}}(v)\,dv \\
&= \frac{e^{-dk}}{\pi}\int_0^\infty e^{-ivk}\psi_T^{\text{MJD, 1}}(v)\,dv \\
&\quad+\frac{e^{-dk}}{\pi}\int_0^\infty e^{-ivk}\left[\psi_T^{\text{MJD, 2}}(v)+\psi_T^{\text{RESIDUAL}}(v)\right]\,dv \\
&:= C_T^{\text{proxy}}(k)+C_T^{\text{residual}}(k).
\end{aligned}
$$

Our Novel Error-Reducing Method

- $$C_T^{\text{proxy}}(k) = \sum_{j=0}^{H-1} \frac{e^{-\lambda' T}(\lambda' T)^j}{j!} C_T^{\text{BS}}(S_0, K, \sigma_j, r_j).$$
- How to determine corresponding parameters?
- $m_1 + \ln S_0$, m_2, m_3, m_4, and m_5: First five cumulants implied by a target process.
- $$(\mu + \lambda\alpha)\, T = m_1, \tag{1}$$
  $$\left(\sigma^2 + \lambda\left(\alpha^2 + \beta^2\right)\right) T = m_2, \tag{2}$$
  $$\lambda(\alpha^3 + 3\alpha\beta^2)\, T = m_3, \tag{3}$$
  $$\lambda(\alpha^4 + 6\alpha^2\beta^2 + 3\beta^4)\, T = m_4, \tag{4}$$
  $$\lambda(\alpha^5 + 10\alpha^3\beta^2 + 15\alpha\beta^4)\, T = m_5. \tag{5}$$

Our Novel Error-Reducing Method

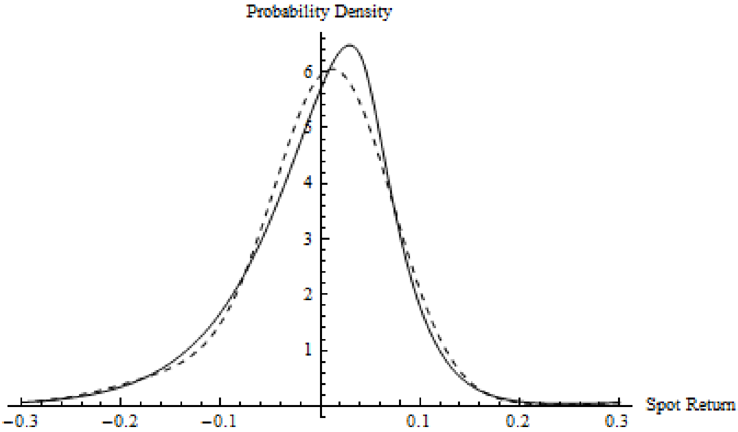
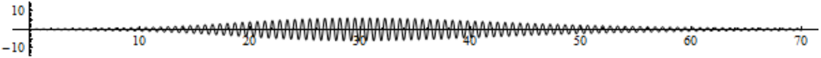


Figure: Comparison of the Target and Proxy Densities

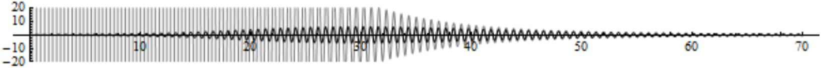
Solid line: Variance Gamma model
Dashed line: Merton's jump-diffusion model

Our Novel Error-Reducing Method

$Re[e^{-ivk}\psi_{\mathrm{VG}}(v)]$

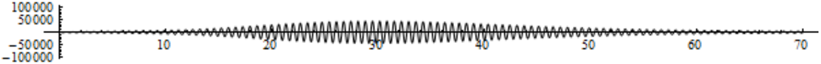


$Re[e^{-ivk}\psi_{\mathrm{residual}}(v)]$

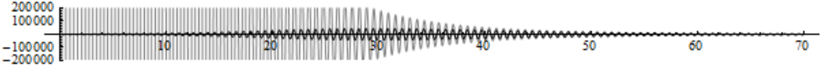


Both

$Re[(e^{-ivk}\psi_{\mathrm{VG}})^{(4)}(v)]$



$Re[(e^{-ivk}\psi_{\mathrm{residual}})^{(4)}(v)]$



Both

Composite Simpson's rule

- A method of numerical integration.
- $f : [a, b] \to \mathbb{R} \in C^4$; n: An even number. There is a $\xi \in [a, b]$ such that

$$\int_a^b f(x)\,dx = \frac{h}{3}[f(x_0)+2\sum_{j=1}^{\frac{n}{2}-1} f(x_{2j})+4\sum_{j=1}^{\frac{n}{2}} f(x_{2j-1})+f(x_n)]-\frac{b-a}{180}h^4 f^{(4)}(\xi),$$

  where $x_j = a + jh$ for $j = 0, 1, ..., n$, and $h = (b-a)/n$.
- The quadrature error is bounded by

$$\frac{b-a}{180}h^4 \sup_{y\in[a,b]} \left|f^{(4)}(y)\right|.$$

Numerical Results

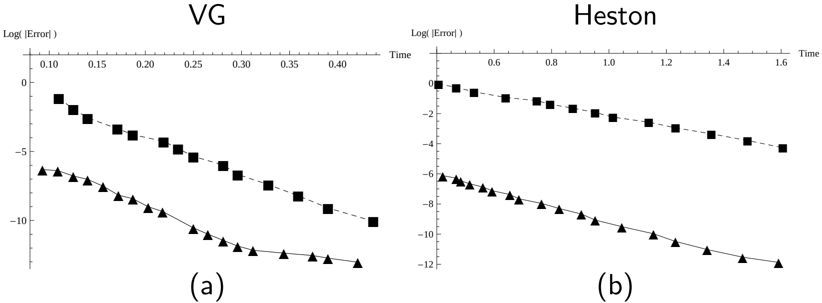


Figure: Convergence of Pricing Results

Square: Carr and Madan's method
Triangle: Carr and Madan's method + Proxy.

Numerical Results

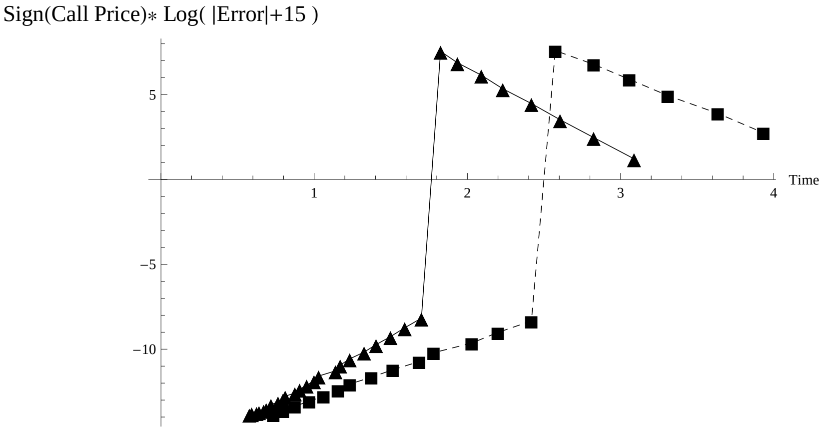


Figure: Comparison of Pricing Results With and Without A Proxy Term.

Square: Carr and Madan's method
Triangle: Carr and Madan's method + Proxy.

Conclusions

- This thesis suggests a novel error-reducing method on the FFT option valuation.
- $C_T(k) = C_T^{\text{proxy}}(k) + C_T^{\text{residual}}(k)$.
- We significantly improve the upper bound of quadrature error. Therefore, given the same computational time limit, our method will have a better chance to generate more accurate results than the original Carr and Madan's method.
- We alleviate the negative pricing results mentioned in Carr and Madan (2009).
- The idea in this paper can also be applied to other kinds of pricing algorithms or payoffs.

Appendix

- We first rewrite Eq. (3) as

$$\beta^2 = \frac{m_3 - \lambda\alpha^3}{3\lambda\alpha} \tag{6}$$

and then replace all the β^2 in Eq. (4) and Eq. (5) by the formula above. This yields

$$-2\alpha^6\lambda^2 + (4m_3\alpha^3 - 3\alpha^2 m_4)\lambda + m_3^2 = 0;$$
$$-2\alpha^6\lambda^2 - 3\alpha m_5\lambda + 5m_3^2 = 0.$$

- By equating the above two equations, we get

$$\lambda = \frac{4m_3^2}{3m_5\alpha - 3m_4\alpha^2 + 4m_3\alpha^3}. \tag{7}$$

- Replacing λ and β^2 in Eq. (3) by Eq. (6) and Eq. (7), respectively, and get a polynomial equation

$$48\alpha^4 m_3^4 - 120\alpha^3 m_3^3 m_4 + 9\alpha^2 m_3^2\left(8m_3 m_5 + 5m_4^2\right)$$
$$-54\alpha m_3^2 m_4 m_5 + 9m_3^2 m_5^2 = 0.$$